On the Determination of Moving Boundaries for Hyperbolic Equations

Gregory Eskin and James Ralston
Department of Mathematics, UCLA

Abstract. We consider wave equations in domains with time-dependent boundaries (moving obstacles) contained in a fixed cylinder for all time. We give sufficient conditions for the determination of the moving boundary from the Cauchy data on part of the boundary of the cylinder. We also study the related problem of accessibility of the moving boundary by time-like curves from the boundary of the cylinder.

§1. Introduction

In this article we study the possibility of determining a moving boundary from Cauchy data on a stationary boundary. The setting of this problem is as follows. Let Q be a connected domain in $\mathbb{R}^n_x \times \mathbb{R}_t$ with smooth boundary ∂Q. We assume that the complement of Q is contained in the cylinder $C = \{(x,t) : |x| < \rho,\ t \in \mathbb{R}\}$, and that ∂Q is moving at speed uniformly less than 1, i.e. if $\nu = (\nu_x, \nu_t)$ is a normal vector to ∂Q, then $|\nu_t| \leq r|\nu_x|$ for a fixed $r < 1$. This condition means that ∂Q is "time-like". These assumptions imply that the sets $\Omega_t = \{x : \ (x,t) \in Q\}$ are diffeomorphic and connected. We think of the complement Ω_t^c as an impermeable obstacle which is smoothly deformed to $\Omega_{t'}^c$ as time goes from t to t'.

The assumption that ∂Q is time-like implies that the following boundary value problems are well-posed for $f \in C_c^\infty(\partial C)$: the forward problem

$$u_{tt} - \Delta u = 0 \text{ in } Q \cap C,\ u = 0 \text{ on } \partial Q,\ u = f \text{ on } \partial C, \text{ and } u = 0 \text{ when } t << 0, \quad (1)$$

and the backward problem

$$u_{tt} - \Delta u = 0 \text{ in } Q \cap C,\ u = 0 \text{ on } \partial Q,\ u = f \text{ on } \partial C, \text{ and } u = 0 \text{ when } t >> 0. \quad (1')$$

Letting u^f denote the solution to the forward problem with boundary data f, we have the set of Cauchy data on ∂C

$$\mathcal{K}(Q) = \{(f, \frac{\partial u^f}{\partial \nu})|_{\partial C} : f \in C_c^\infty(\partial C)\}.$$

With these definitions one can ask the question: does $\mathcal{K}(Q)$ determine Q?

Surprisingly, even when the motion is periodic, i.e. when Q is invariant under the mapping $t \to t+1$, the answer is no. This was discovered by Stefanov in [St]. So the general problem is to characterize the Q which are determined by $\mathcal{K}(Q)$. This still appears quite difficult. Here we will restrict ourselves to a discussion of sufficient conditions for the Cauchy data to determine Q.

A condition more easily verified than $\mathcal{K}(Q)$ determines Q is that all points of ∂Q are accessible both forward and backward in time. We define $(x_0, t_0) \in \partial Q$ to be accessible (forward) if there is a piecewise differentiable curve $(x(t), t)$ in Q

with $|\dot{x}(t)| \leq 1$ such that $x(t_0) = x_0$ and $(x(t_1), t_1) \in C$ for some $t_1 < t_0$. Accessible backward is defined the same way with $t_1 > t_0$. The failure of accessibility is essential in Stefanov's example. In §2 we discuss accessibility in two space dimensions. We also show that all points will be accessible if one can parameterize ∂Q in the following way: suppose that $(x,t) \in \partial Q$ is equivalent to $x = \phi(y,t)$, where $\phi(\cdot,t)$ is a smooth diffeomorphism of $\partial\Omega_0$ into $\mathbb{R}^n$ satisfying $\phi(y,t) = \phi(y,t+1)$ and $|\phi_t(y,t)| < 1$. In other words ∂Q is moving periodically, and has a periodic parametrization with time derivative having norm less than one (see Proposition 2.2). Note that the assumption that ∂Q is moving at speed less than one only implies that the normal component of the time derivative of ϕ has norm less than one. In §3 we present a simplified version of Stefanov's example.

An implicit sufficient condition for $\mathcal{K}(Q)$ to determine Q is that there is a diffeomorphism, $\Phi(y,t) = (\Psi(y,t),t)$ such that $\Psi(y,t) = y$ on ∂C, $\Psi(\cdot,T)$ maps $Q \cap C \cap \{t = 0\}$ onto $Q \cap C \cap \{t = T\}$ for all T, and satisfies the following (surprisingly restrictive) conditions: $|\Psi_t(y,t)| < 1$ and $||\Psi_y(y,t)|| \leq M < \infty$ for all (y,t). Here Ψ_y is the Jacobian matrix. We give the proof of this result in §4 for more general hyperbolic operators of the form

$$\partial_t^2 u - 2a(t,x) \cdot \nabla_x \partial_t u - \nabla_x \cdot A(t,x) \nabla_x u - L_1(t,x,\partial_t,\partial_x)u, \tag{2}$$

where A is positive definite and L_1 is a differential operator of order one. We also discuss four situations where one can construct Ψ with the properties mentioned above.

There is substantial literature on wave equations in domains with moving boundaries. In particular, the fundamental existence and uniqueness results were established by Ikawa [I], and Cooper and Strauss developed scattering theory for these equations and solved the inverse problem for moving convex boundaries in a series of papers (see [CS1-3]). Their proof also shows that $\mathcal{K}(Q)$ determines Q when $\partial\Omega_t$ is convex for all t

§2. **Periodic Obstacles in Two Space Dimensions: Accessibility**

In this section we consider domains $Q \subset \mathbb{R}_x^2 \times \mathbb{R}_t$ with boundary ∂Q given by $(x(\sigma,t),t)$, $\sigma \in S^1$. We assume that $x(\sigma,t)$ is smooth, non-degenerate: $|x_\sigma| > 0$, and periodic: $x(\sigma, t+T) = x(\sigma,t)$. The boundary of Q will be time-like for $u_{tt} - \Delta$ precisely when the normal component of x_t is strictly less than one, i.e. when $|x_t - (x_t \cdot x_\sigma)|x_\sigma|^{-2} x_\sigma| < 1$. This condition can be stated in several equivalent forms. One that is useful is

$$|x_\sigma|^2 |x_t|^2 < (x_t \cdot x_\sigma)^2 + |x_\sigma|^2. \tag{3}$$

Instead of taking $\sigma \in S^1$, it is convenient to use the equivalent formulation $\sigma \in \mathbb{R}$ with $x(\sigma+1,t) = x(\sigma,t)$. Note that the periodicity in both t and σ implies that the non-degeneracy and time-like conditions hold uniformly: $|x_\sigma|^2 \geq \delta > 0$ and $|x_\sigma|^2 + (x_t \cdot x_\sigma)^2 - |x_\sigma|^2 |x_t|^2 \geq \delta > 0$ for all (σ,t) for some $\delta > 0$.

Suppose that $x(\sigma(t),t)$ is a null geodesic (light curve) in the Minkowski metric $(dt)^2 - (dx_1)^2 - (dx_2)^2$ restricted to ∂Q. Then

$$0 = 1 - |x_t|^2 - 2(x_\sigma \cdot x_t)\sigma' - |x_\sigma|^2 (\sigma')^2,$$

and we have two differential equations for possible $\sigma(t)$'s

$$\frac{d\sigma_\pm}{dt} = \Lambda_\pm(\sigma, t), \text{ where}$$

$$\Lambda_\pm = \frac{-x_\sigma \cdot x_t \pm \sqrt{(x_\sigma \cdot x_t)^2 + (1 - |x_t|^2)|x_\sigma|^2}}{|x_\sigma|^2} \tag{4}$$

Note that $\Lambda_\pm$ are real by (3).

We would like to have (fairly) sharp conditions for the existence of time-like curves connecting points in $Q \cap C$ to ∂C (accessibility). For definiteness we will only consider accessibility *forward* in time here. The analogs of our results for accessibility backward in time will be obvious. Our first result is:

Proposition 2.1. If any point on ∂Q is accessible from ∂C, then any point in $Q \cap C$ is accessible from ∂C.

Proof. Suppose that (x_0, t_0) is in the interior of Q. Since $\Omega_{t_0} = Q \cap \{t = t_0\}$ is a connected set with smooth boundary, we can choose a simple path $x(\sigma)$ parameterized by arc length with $x(0) = x_0$ and $x(l) \in \partial\Omega_{t_0}$ such that $x(\sigma)$ is in the interior of Ω_{t_0} for $\sigma < l$. Let

$$\sigma_0 = \inf\{\sigma \in [0, l] : \cup_{t \in \mathbb{R}}(x(\sigma), t) \cap \partial Q \neq \emptyset\}$$

If $\sigma_0 = 0$, we can just follow the line (x_0, t) until it hits ∂Q. If $\sigma_0 > 0$ we proceed as follows. Since the motion of the boundary is periodic, the vertical line $(x(\sigma_0), t)$ intersects ∂Q at a sequence of points $(x(\sigma_0), t_1 + nT),\ n \in \mathbb{Z}$. Since $\{(x(\sigma), t) : \sigma \in [0, \sigma_0),\ t \in \mathbb{R}\}$ is in the interior of Q, the path $(x(\sigma), t_0 - 2\sigma),\ 0 \leq \sigma \leq \sigma_0$, will be time-like and connect $(x(\sigma_0), t_0 - 2\sigma_0)$ to (x_0, t_0) in Q. Choose n so that $t_n = t_1 - nT \leq t_0 - 2\sigma_0$. Since $(x(\sigma_0), t_n)$ can be connected to ∂C by a time-like path by hypothesis and $(x(\sigma_0), t),\ t_n \leq t \leq t_0 - 2\sigma_0$ is time-like (note that $|x'(\sigma)| = 1$), we see that (x_0, t_0) is accessible from ∂C.

In view of Proposition 2.1 we would like to find conditions implying that all points on ∂Q are accessible from ∂C. There are always some accessible points on ∂Q: if x_0 is a point on $\partial\Omega_{t_0}$ such that $|x|$ is maximal, then the lines $(x_0 \pm (t - t_0)x_0/|x_0|, t - t_0)$ lie in Q for $t > t_0$ and $t < t_0$ respectively, since the ∂Q is time-like. Hence we have a sequence of points $(x_0, t_n) = (x_0, t_0 + nT),\ n \in \mathbb{Z}$, on ∂Q which are accessible. So we will look for conditions implying that arbitrary points on ∂Q can be reached from these points by time-like curves *lying in* ∂Q. Our next result is

Proposition 2.2. If $\min \Lambda_+ > \max \Lambda_-$, then all points in ∂Q are accessible in ∂Q from the points (x_0, t_n).

Proof. For $\alpha \in [0, 1]$ let $\sigma_\alpha(t)$ be the solution to

$$\sigma' = \alpha\Lambda_-(\sigma, t) + (1 - \alpha)\Lambda_+(\sigma, t),\ \sigma(t_0) = \sigma_0,$$

where $x(\sigma_0, t_0) = x_0$. Then $(x(\sigma_\alpha(t), t), t)$ is time-like for $\alpha \in [0, 1]$ and $\sigma_\alpha(t)$ depends continuously on α. Assuming that $\sigma_\pm(t)$ have the initial data $\sigma_\pm(t_0) = \sigma_0$, we have $\sigma_\alpha(t) = \sigma_-(t)$ when $\alpha = 1$ and $\sigma_\alpha(t) = \sigma_+(t)$ when $\alpha = 0$. Hence, it follows by the intermediate value theorem that $\sigma_\alpha(t)$ takes all values between $\sigma_-(t)$

and $\sigma_+(t)$ as α goes from 0 to 1. Thus all of the points $(x(\sigma,t),t),\ \sigma_-(t)<\sigma<\sigma_+(t),\ t>t_0$ are accessible from (x_0,t_0).

By the mean value theorem for $t \geq t_0$

$$\sigma_+(t)-\sigma_-(t)=(t-t_0)(\Lambda_+(\sigma_+(t^*),t^*)-\Lambda_-(\sigma_-(t^*),t^*))$$

for some t^* between t_0 and t. Thus by the hypothesis there is a $\delta > 0$ such that $\sigma_+(t)-\sigma_-(t)\geq \delta(t-t_0)$. Hence, in view of the periodicity of $x(\sigma,t)$ in σ, there is a t_1 such that all points on $\{(x,t)\in\partial Q : t\geq t_1\}$ are accessible from (x_0,t_0). Using the periodicity of $x(\sigma,t)$ in t, it follows that any point on ∂Q can be reached from one of the points (x_0,t_0+nT).

One could conjecture that if the curves $\sigma_+(t)$ and $\sigma_-(t)$ starting from (σ_0,t_0) are both unbounded, then all points on ∂Q with t sufficiently large will be accessible. Unfortunately, this is not always the case: it is easy to construct examples (with $\min\Lambda_+ = \max\Lambda_-$) where these curves follow each other so closely that only a small subset of ∂Q is accessible from $(x(\sigma_0,t_0),t_0)$. We have not found a truly sharp hypothesis for accessibility.

The requirement that the motion be periodic forces the curves $\sigma_\pm(t)$ to either be unbounded or asymptotic to periodic orbits as $t\to\pm\infty$. The precise statement is the following proposition.

Proposition 2.3. Assume for simplicity that $T=1$. Let $\sigma(t)$ be a solution to either $\sigma'=\Lambda_+(\sigma,t)$ or $\sigma'=\Lambda_-(\sigma,t)$ for $t\in\mathbb{R}$. If $\sigma(1)>\sigma(0)$ and $\sigma(t)$ is bounded above as $t\to\infty$, then $\sigma(t)$ is asymptotic from below to a periodic orbit. If $\sigma(1)<\sigma(0)$ and $\sigma(t)$ is bounded below as $t\to\infty$, then $\sigma(t)$ is asymptotic from above to a periodic orbit. If $\sigma(1)=\sigma(0)$, then $\sigma(t)$ is periodic.

Proof. Consider the case when $\sigma(1)<\sigma(0)$ and $\sigma(t)\geq\sigma_0>-\infty$ for $t\geq 0$. Since $\sigma(t+1)$ is also a solution of the equation, and the solution passing through a point (t_0,σ_0) is unique, it follows that $\sigma(t)>\sigma(t+1)$ for all t. Repeating this argument one sees that $\sigma(t)>\sigma(t+1)>\sigma(t+2)>\cdots$ for all t. Defining $w_n(t)=\sigma(t+n)$ we have decreasing sequence of solutions bounded below by σ_0. So $\lim_{n\to\infty} w_n(t)=w_\infty(t)$ exists for all $t\geq 0$. Since, letting F denote Λ_+ or Λ_-, we have

$$w_n(t)=w_n(0)+\int_0^t F(s,w_n(s))ds,$$

the Arzela-Ascoli theorem implies that the convergence of a subsequence to $w_\infty(t)$ is uniform on bounded intervals, and hence that $w_\infty(t)$ is also a solution of $\sigma'=F(\sigma,t)$. Since

$$w_\infty(0)=\lim_{n\to\infty}\sigma(n)=\lim_{n\to\infty}\sigma(n+1)=w_\infty(1),$$

$w_\infty(t)$ is a periodic solution. Dini's theorem implies that the convergence of $w_n(t)$ to $w_\infty(t)$ is uniform on $[0,1]$, and for $t\in[0,1]$ we have

$$|w_\infty(t+n)-\sigma(t+n)|=|w_\infty(t)-w_n(t)|<\epsilon$$

when $n\geq N(\epsilon)$. Hence $\sigma(t)$ is asymptotic to $w_\infty(t)$ as $t\to\infty$. The proof for the case $\sigma(1)>\sigma(0)$ is the same, using increasing sequences in place of decreasing sequences.

A consequence of Proposition 2.3 is that, when $\sigma_+(t)$ with $\sigma_+(t_0) = \sigma_0$ is bounded above and $\sigma_-(t)$ with $\sigma_-(t_0) = \sigma_0$ is bounded below, there will be two periodic orbits which may make part of ∂Q inaccessible (forward and backward) from (σ_0, t_0) along time-like curves in ∂Q. That is what happens in Stefanov's example.

§3. Stefanov's Example Revisited

Consider a domain with part of its boundary given by

$$(x_1(\sigma,t), x_2(\sigma,t)) = (\sigma, \phi(\sigma)f(k(2\sigma - t))),\ |\sigma| < M + L,\ k \in \mathbb{N}$$

Here $|f| \leq 1$ and f is function of period one, $\phi \in C_c^\infty(|\sigma| < M + L)$ and $\phi(\sigma) = 1$ for $|\sigma| \leq M$. In this construction it would suffice to have $M = L = 2$, but we have kept the notation M and L to distinguish the supports of ϕ and ϕ'.

To compute the normal component of x_t we note

$$x_t = (0, k\phi f') \text{ and } x_\sigma = (1, \phi' f + 2k\phi f').$$

Hence, $|\nu_x \cdot x_t| = |a|(1 + (b + 2a)^2)^{-1/2}$, where $a = k\phi f'$ and $b = \phi' f$. From this one can show that $|x_t \cdot \nu_x| \leq 1/\sqrt{2}$, when $|b| \leq 1$. Since $|f| \leq 1$, it suffices to have $|\phi'| \leq 1$, and one can arrange that when $L > 1$. Thus with these choices this portion of ∂Q is time-like.

For this boundary the equation for $\sigma_-(t)$ when $|\sigma_-(t)| \leq M$ is

$$\sigma_-' = \Lambda_-(\sigma_-, t) = \frac{2k^2(f')^2 - \sqrt{1 + 2k^2(f')^2}}{1 + 4k^2(f')^2} \geq -1.$$

From that one can see that it is going to be very difficult for $\sigma_-(t)$ to move to the left. Assume that $\sigma_-(t_0) = 0$.

To check that $\sigma_-(t)$ cannot move very far to the left, define $w(t) = 2\sigma_-(t) - t$. Then $w(t)$ satisfies the autonomous equation

$$w' = \frac{-1 - 2\sqrt{1 + 2k^2(f'(kw))^2}}{1 + 4k^2(f'(kw))^2} =_{\text{def}} -F(kw, k).$$

and

$$t - t_0 = \int_{w(t)}^{w(t_0)} \frac{dw}{F(kw,k)} = \frac{1}{k}\int_{kw(t)}^{kw(t_0)} \frac{dz}{F(z,k)}.$$

For a function of period 1, writing $b - a = m + r$ with $m \in \mathbb{N}$ and $0 \leq r < 1$, one has

$$\int_a^b f dz = (b - a)\int_0^1 f dz - r\int_0^1 f dz + \int_a^{a+r} f dz.$$

Applying this with $f = (kF(z,k))^{-1}$ gives

$$t - t_0 = (w(t_0) - w(t))H_0 - \frac{r}{k}H_0 + \frac{1}{k}\int_{kw(t)}^{kw(t)+r} \frac{dz}{F(z,k)},$$

where $0 \le r < 1$ and $H_0 = \int_0^1 dz/F(z,k)$. Substituting $2\sigma_-(t) - t$ for $w(t)$ and solving gives

$$2\sigma_-(t) = (1 - \frac{1}{H_0})(t - t_0) - \frac{r}{k} + \frac{1}{kH_0}\int_{kw(t)}^{kw(t)+r} \frac{dz}{F(z,k)} \ge (1 - \frac{1}{H_0})(t - t_0) - \frac{1}{k},$$

since $F(z,k)$ does not change sign. Thus, assuming $M > 1/2$, $\sigma_-(t)$ will never reach $-M$, if $H_0 > 1$. We have

$$H_0 =$$

$$\int_0^1 \frac{1 + 4k^2(f'(z))^2}{1 + 2\sqrt{1 + 2k^2(f'(z))^2}} dz > \int_0^1 \frac{1}{2}\sqrt{1 + 2k^2(f'(z))^2} dz > \frac{k}{\sqrt{2}} \int_0^1 |f'(z)| dz.$$

So $|H_0| > 1$ when $k\int_0^1 |f'(z)|dz > \sqrt{2}$. This determines our choice of k and shows that for this example points on ∂Q with $\sigma = -M$ are inaccessible (forward) in the boundary from points with $\sigma = M$.

To turn this into an example of a domain $Q \subset \mathbb{R}^2_x \times \mathbb{R}_t$ with points on ∂Q inaccessible from ∂C, we need to specify the rest of ∂Q in a way that forces any curve in Q reaching $x = (M + L, 0)$ to follow the boundary constructed above very closely. Here we use the original idea in [St]: we add another boundary curve below, but very close to the boundary just constructed, so that any curve in Q reaching $x = (-M - L, 0)$ must pass through the narrow "channel" between these curves.

Let $\nu(\sigma, t)$ be the unit normal (directed downward) to the curve $x(\sigma, t)$ and consider

$$x(\sigma, t, \eta) = x(\sigma, t) + \eta\nu(\sigma, t)$$

Since the curvature of the curve traced by $x(\cdot, t)$ is bounded for all t, there is a $\delta > 0$ such that (σ, η) are coordinates on $\{x : |x_1| < M+L,\ x_2(x_1, t) - \epsilon < x_2 \le x_2(x_1, t)\}$ for ϵ sufficiently small. We are going to take $x_2 = x_2(x_1, t) - \epsilon$ as the boundary of the lower side of the channel, but we will be taking ϵ smaller later in the argument.

A curve of speed less than one in the channel $x(\sigma(t), t, \eta(t))$ satisfies

$$1 \ge |\dot\sigma x_\sigma + x_t + \dot\eta\nu + \eta\dot\nu|^2$$

$$= |x_\sigma|^2\dot\sigma^2 + |x_t|^2 + \dot\eta^2 + \eta^2|\dot\nu|^2 + 2\dot\sigma x_\sigma \cdot x_t + 2\eta\dot\sigma x_\sigma \cdot \dot\nu + 2\dot\eta\nu \cdot x_t + 2\eta x_t \cdot \dot\nu, \quad (5)$$

where we used $\nu \cdot x_\sigma = \nu \cdot \dot\nu = 0$ since ν is the unit normal. Since $|\nu \cdot x_t| < 1$, we have $\dot\eta^2 + 2\dot\eta\nu \cdot x_t + 1 > 0$ and can rewrite (5) as

$$2 \ge |x_\sigma|^2\dot\sigma^2 + |x_t|^2 + +2\dot\sigma x_\sigma \cdot x_t - O(\eta), \quad (6)$$

where the $O(\eta)$ term does not depend on $\dot\eta$. Solving (6) for $\dot\sigma$ gives

$$\tilde\Lambda_-(\sigma, t) \le \dot\sigma \le \tilde\Lambda_+(\sigma, t),\ \tilde\Lambda_\pm = \frac{-x_\sigma \cdot x_t \pm \sqrt{(x_\sigma \cdot x_t)^2 + (2 + O(\eta) - |x_t|^2)|x_\sigma|^2}}{|x_\sigma|^2}.$$

Now suppose that the $O(\eta)$ term is less than 1. Then the argument given earlier shows that $\sigma(t)$ cannot reach $-M$ for any $t > 0$ when k is sufficiently large (in this case it suffices to have $k\int_0^1 |f'(z)|dz > \sqrt{6}$). Having chosen that k we then take δ sufficiently small that the $O(\eta)$ term is less than 1, and finally choose ϵ so that

the channel is entirely in the region where $0 \le \eta < \delta$. Thus no time-like curve can pass through the channel and the points in Q with $x_1 = -M - L, -\epsilon < x_2 < 0$ are inaccessible (forward) for all time.

Remark. In [St] Stefanov assumed that the rest of ∂Q was defined so that the intersection of the cylinder $\{|x-(-M-L,0)| < 2M+2L\} \times \mathbb{R}_t$ with Q only contains the channel. Then one can use the domain of dependence theorem of Inoue [In] to show that, if a solution to the forward problem were nonzero near $x = (-M-L, 0)$ at some time, then there would necessarily be a time-like path through the channel. Hence, by contradiction, $((-M-L,0),t)$ is outside the domain of dependence of ∂C for all t.

§4. Determination of Q from Cauchy Data for General Hyperbolic Equations

§4.1. In this section we consider the more general hyperbolic equation $Lu = 0$ from the Introduction with

$$Lu = \partial_t^2 u - 2a(t,x) \cdot \nabla_x \partial_t u - \nabla_x \cdot A(t,x) \nabla_x u - L_1(t,x,\partial_t,\partial_x)u,$$

where L_1 is a differential operator of order one. All coefficients are assumed to be real, bounded and smooth on $\mathbb{R}_t \times \mathbb{R}_x^n$, and real analytic in t. L is strictly hyperbolic with respect to t if for $\xi \neq 0$

$$0 = p_2(x,t,\xi,\tau) =_{\text{def.}} \tau^2 - 2\tau a \cdot \xi - \xi \cdot A\xi$$

has distinct real roots

$$\tau_\pm(x,t,\xi) = a \cdot \xi \pm \sqrt{(a \cdot \xi)^2 + \xi \cdot A\xi}.$$

We make the stronger hypothesis that $A(x,t) \ge \delta I,\ \delta > 0$, for all $(t,x) \in \mathbb{R}_x^n \times \mathbb{R}_t$. Hence τ_+ and τ_- have opposite signs. This has the following interpretation in pseudo-riemannian geometry. Writing $p_2(x,t,\xi,\tau)$ as a quadratic form $(\xi,\tau) \cdot B(x,t)(\xi,\tau)$, the dual form on tangent vectors is $(v_x,v_t) \cdot B^{-1}(x,t)(v_x,v_t)$. One says that a curve in space-time is "time-like" if its tangent vector $v = (v_x, v_t t)$ satisfies $v \cdot B^{-1}(x,t)v > 0$. The hypothesis $A > 0$ is equivalent to assuming that the time curves $\gamma(t) = (x_0,t)$ are time-like for L (see §4.2).

We consider solutions of $Lu = 0$ in a time-dependent (open) domain Q in $\mathbb{R}_x^n \times \mathbb{R}_t$ where the boundary ∂Q is smooth and uniformly time-like for L, i.e. there is a $\delta > 0$ such that $p_2(x,t,\xi,\tau) \le -\delta|(\xi,\tau)|^2$ when (ξ,τ) is normal to ∂Q at (x,t). We also assume that for each $t \in \mathbb{R}$ the set $\Omega_t = \{x : (x,t) \in Q\}$ is a connected exterior domain in $\mathbb{R}^n$, and the complement of Q is contained in the cylinder $C = \{(t,x) : |x| < \rho\}$. For positive results we will need the following additional hypothesis on Q. We assume that we have diffeomorphisms $\Psi^t(y)$ with the following properties:

(i) for each t, $\Psi^t(y)$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$ taking Ω_0 onto Ω_t (and $\partial\Omega_0$ onto $\partial\Omega_t$),

(ii)$\Psi^0(y) = y,\ y \in \Omega_0$, and $\Psi^t(y) = y$ near ∂C for all $t \in \mathbb{R}_t$,

(iii) $(\Psi^t(y), t)$ is time-like for L for all $y \in \Omega_0$. We assume that this holds uniformly in the sense that $(\partial_t \Psi^t, 1) \cdot B^{-1}(\Psi^t, t)(\partial_t \Psi^t, 1) \le -\delta < 0$ for $(y,t) \in \Omega_0 \times \mathbb{R}_t$ and

(iv) the Jacobian matrix of Ψ^t satisfies $||\partial_y \Psi^t(y)|| \leq M < \infty$ for all $(y,t) \in \Omega_0 \times \mathbb{R}_t$.

In [St] Stefanov gave a construction for $L = \partial_t^2 - \Delta$ of Ψ^t satisfying (i)-(iii) for *any* Q with time-like boundary and complement in C as the solution of $\dot{x} = v(x,t)$, $x(0,y) = y$ for a suitable vector field $v(x,t)$ on Q, tangent to ∂Q. This can be generalized to the setting here (see §4.2). So the additional hypothesis here is really (iv). Now we can state

Theorem 4.1. Suppose that R is another connected domain with time-like boundary and complement contained in C. Let Γ be an open subset of $\{|x| = \rho\}$. If the Cauchy data $\mathcal{K}(Q)$ and $\mathcal{K}(R)$ are identical on $\Gamma \times \mathbb{R}_t$, then $R = Q$.

Proof. We will work in the domain $\hat{Q} = \Omega_0 \times \mathbb{R}_t$, pulling L back to $\hat{L}$ on this domain by the mapping $(x,t) = \Phi(y,t) = (\Psi^t(y), t)$.

Suppose that R is a second domain as in the statement of the theorem. If $R \neq Q$, then there will be either boundary points of R in the interior of Q or boundary points of Q in the interior of R. We begin with (x_0, t_0) which is a boundary point of R in the interior of Q and let (y_0, t_0) be its pre-image under Φ. By the assumption that Ω_{t_0} is connected, we can choose a smooth, non-self-intersecting path $y(\sigma)$, $0 \leq \sigma \leq l$ in Ω_{t_0}, such that $|y'(\sigma)| = 1$, $y(0) = y_0$ and $y(l) \in \Gamma$. For convenience later we choose $y(\sigma)$ so that $y'(l)$ is radial and hence normal to ∂C.

Now we would like to choose a constant $a > 0$ so that $\gamma_+(t) = (y(a(t - t_0)), t)$ is time-like for $\hat{L}$ for $t_0 \leq t \leq t_0 + l/a$. Hence we want $\dot{\gamma}_+ \cdot \hat{B}^{-1} \dot{\gamma}_+ > 0$, where $\hat{B}$ is the quadratic form associated with $\hat{L}$. Note that

$$\hat{B}^{-1}(y,t) = (\partial_{y,t}\Phi(y,t))^T B^{-1}(\Phi(y,t))(\partial_{y,t}\Phi(y,t)).$$

Thus hypotheses (iii) and (iv) imply that

$$|(ay'(\sigma)), 1) \cdot \hat{B}^{-1}(y,t)(ay'(\sigma), 1) - (0,1) \cdot \hat{B}^{-1}(0,1)| \leq Ca$$

uniformly for $(y,t) \in \hat{Q}$ and $\sigma \in [0,l]$. This is the crucial use of (iv): there is no choice of Φ which will make this estimate true in Stefanov's example. Since hypothesis (iii) also implies that the vector $(v_x, v_t) = (0,1)$ is uniformly time-like for $\hat{L}$, we can choose the constant a so that γ_+ is time-like. Likewise, $\gamma_-(t) = (y(-a(t - t_0)), t)$ is time-like for $t_0 - l/a \leq t \leq t_0$.

Consider the two dimensional surface S_0 given by

$$S_0 = \{(y(a(t - t_0)), s),\ |s - t_0| \leq |t - t_0| \leq l/a\}.$$

S_0 is roughly triangular with boundary curves γ_+, γ_- and $\gamma_0(t) = (y(l), t))$, $|t - t_0| \leq l/a$. Clearly with our hypotheses it possible that $S_0 \cap \partial\hat{R} \neq \emptyset$, where $\hat{R}$ is the pull-back of R under Ψ. However, since S_0 is compact and the points on S_0 could be parameterized by (σ, s), where $\sigma = a|t - t_0|$, we can choose $(y_1, t_1) \in S_0 \cap \partial\hat{R}$ where σ assumes its maximum, σ_1. Then we repeat the construction of S_0, using t_1 in place of t_0, $y(a(t - t_1) + \sigma_1)$ in place of $y(a(t - t_0))$, and $y(-a(t - t_1) + \sigma_1)$ in place of $y(-a(t - t_0))$. Note that we can take the new S_0 to be a subset of the original S_0. After this correction, we go back to the original notation, relabeling (y_1, t_1) as (y_0, t_0). Hence we can now assume that S_0 is contained in the interior

of $\hat{Q}$ and intersects the boundary of $\hat{R}$ only at (y_0, t_0). Note also that, since $\partial\hat{R}$ is time-like, taking a slightly *larger* we can assume that γ_+ and γ_- are not tangent to $\partial\hat{R}$ at (y_0, t_0). In what follows we will consider the surface $\Sigma_\epsilon = \partial\hat{R} \cap \{(y,t) : |(y - y_0, t - t_0)| < \epsilon\}$ with ϵ small enough that Σ_ϵ intersects the sphere of radius ϵ around (y_0, t_0) transversally.

In this proof we will use recent extensions of Holmgren's uniqueness theorem to reach a contradiction to the existence of (x_0, t_0). This requires an increasing sequence of domains in $\hat{Q} \cap \hat{R}$ with smooth time-like boundaries which connect $\Gamma \times \mathbb{R}_t$ to a domain whose boundary contains a portion of Σ_ϵ. In what follows we construct such a sequence so that the boundary of the final domain intersects $\partial\hat{R}$ only at (x_0, t_0). However, it will be clear that we could have performed the same construction with (x_0, t_0) replaced by any nearby point on Σ_ϵ. This will give us a sufficiently large set of domains in which to use the uniqueness theorem, and reach a contradiction.

We begin the construction by introducing two dimensional surfaces S_σ contained in S_0

$$S_\sigma = \{(y(a(\sigma)(t - t_0)) + \sigma, s),\ |s - t_0| \le |t - t_0| \le l/a\},\ 0 < \sigma \le l,$$

where $a(\sigma) = (1 - \sigma/l)a$. Note that, since $0 \le a(\sigma) \le a$, the upper and lower boundary curves of S_σ, $\gamma_\pm^\sigma$, are time-like. The surfaces S_σ share the boundary curve γ_0 in ∂C, and have the points $(y(\sigma), t_0)$, $0 \le \sigma \le l$, as "vertices". Note that the surfaces S_σ are nested: $S_\sigma \subset S_{\sigma'}$ when $\sigma > \sigma'$.

Next, we consider the bounded regions $D_\pm$ bounded by ∂C and the parametric surfaces $B_\pm$ defined by

$$\{(y(s), t_0 \pm s/a) + r(s)\omega : 0 < s < l,\ \omega \in \mathbb{S}^n \cap \pi(s)\},$$

where $\pi(s)$ is the plane through $(y(s), t_0)$ perpendicular to $(y'(s), 0)$. Hence B_+ and B_- are unions of $(n-1)$-dimensional spheres in the n-dimensional planes perpendicular to the curve $(y(s), 0)$, $0 < s \le l$, of varying radii with centers on γ_+ and γ_- respectively. In order for these regions to lie in $\hat{Q} \cap \hat{R}$ and have smooth boundary $r(s)$ must be small. We choose $r(s)$ small, tending to zero as s goes to zero, and increasing monotonically with s. However, we keep it small enough that $D_\pm \cap \partial C \subset \Gamma \times \mathbb{R}_t$.

Next we take D_0 to be the union of the convex hulls of the intersections of $D_\pm$ with the planes $\pi(s)$ for $0 < s \le l$. In other words D_0 is the union of a family of "stadium domains", the convex hulls of sets consisting of two spheres. The part of ∂D_0 which is not in $B_- \cup B_+$ consists of vertical lines ("vertical" means parallel to the t-axis) and hence is time-like. Any point in the boundary of D_0 which is in $B_+ \cup B_-$ must have the form given in the parametrization above with the t component of ω nonnegative on B_+ and nonpositive on B_-. Hence, taking the curve though such a point varying s in the parametrization and holding ω constant, one gets a curve with a time-like tangent. Hence the boundary of $D_0 \cap C$ is time-like (see §4.2 for a proof of the basic result that a hypersurface containing a time-like curve will be time-like at all points on that curve).

To construct an exhaustion of D_0 by increasing domains with time-like boundaries we repeat the preceding replacing, $\gamma_\pm$ by $\gamma_\pm^\sigma$. We construct $B_\pm^\sigma$ as the union of spheres in the space-time planes perpendicular to $(y(s), t_0)$, $\sigma \le s \le l$ with

the same $r(s)$ as before. However, when we take the convex hulls of the regions bounded by these surfaces in the planes $\pi(s)$, they end in the disk of radius $r(\sigma)$ centered at $(y(\sigma), t_0)$ in the plane perpendicular to $(y'(\sigma), 0)$. However, we know that the tangent planes to the boundary of this set remain strictly time-like when one approaches the disk from $s > \sigma$. Hence there is no difficulty building D_σ by adding a "cap" to the region ending in the disk so that D_σ time-like boundary and contains $(y(s), t_0)$ for $\sigma > s > \sigma - \delta(\sigma) > 0$. Moreover, one can add the caps to that D_σ is monotonically decreasing with σ.

The boundaries of the domains D_σ are C^1, failing to be smooth where the vertical lines from the convex hulls touch $B^\sigma_\pm$. However, one can smooth them near these points preserving the time-like boundaries and monotonicity of the D_σ's.

To construct the domain D_0 and its exhaustion in the case that (y_0, t_0) is a boundary point of $\hat{Q}$ in the interior of $\hat{R}$, one proceeds in the same way, defining $\Sigma_\epsilon = \partial\hat{Q} \cap \{|(y,t) : |(y - y_0, t - t_0)| < \epsilon\}$. Note that the only case of $R \neq Q$ where there will be no interior points of Q which are boundary points of R is $Q \subset R$. Hence without loss of generality we can assume that $S_0 \cap \partial\hat{R} = \emptyset$ here, omitting the consideration of $(x_1, t_1) \in S_0 \cap \partial\hat{R}$ required in the preceding case.

This is the construction needed for the use unique continuation theorems in the case that $\partial D_0 \cap \partial\hat{R} = \{(x_0, t_0)\}$. For $(x', t') \in \Sigma_\epsilon$ sufficiently close to (x_0, t_0) one can repeat the construction replacing x_0 by x' and t_0 by t' at all places where they appear. There is one slightly subtle point in this argument: in order to replace S_0 by a small perturbation S' without hitting new points of $\partial\hat{R}$ the condition that $\partial\hat{R}$ is not tangent to $\gamma_\pm$ at (x_0, t_0) must be used. We denote the domain D_0 ending at (x', t') by $D_0(x', t')$.

Our assumption $\mathcal{K}(R) = \mathcal{K}(Q)$ on $\Gamma \times \mathbb{R}_t$ implies that solutions to the forward problem in $Q \cap C$

$$Lu = 0 \text{ in } Q \cap C,\ u = 0 \text{ on } \partial Q,\ u = f \text{ on } \partial C, \text{ and } u = 0 \text{ when } t << 0,$$

and the same problem with Q replaced by R, have the same Cauchy data on $\Gamma \times \mathbb{R}_t$. This will make them identical on the sets $\Phi(D_0(x', t'))$. To prove that note first that, since the boundaries of the D_σ's are time-like for $\hat{L}$, the boundaries of their images under Φ are time-like for L. Thus we have an exhaustion of $\Phi(D_0(x', t'))$ regions with time-like boundaries which intersect ∂C in fixed subset of $\Gamma \times \mathbb{R}_t$. Thus, assuming that $u^f_Q - u^f_R$ does not vanish identically in $\Phi(D_0(x', t'))$, there is a last σ such that it vanishes on $D_\sigma(x', t')$. Since we have assumed that the coefficients of L are analytic in t, the unique continuation theorems of Robbiano-Zuily and Tataru (see [RZ] and [T]) give a contradiction, and we conclude that $u^f_Q - u^f_R$ vanishes on $\Phi(D_0(x', t'))$. Note that for $L = \partial_t^2 - \Delta$ this step only requires Holmgren's theorem. Since we have this conclusion for all (x', t') in a neighborhood Σ of (x_0, t_0) in Σ_ϵ (meaning $(x_0, t_0) \in \Sigma \subset \Sigma_\epsilon$), we conclude in conclusion that $u^f_Q - u^f_R$ vanishes on

$$D = \cup_{(x', t') \in \Sigma} D_0(x', t').$$

Let $G_Q(x, z, t, s)$ and $G_R(x, z, t, s)$ be the backward fundamental solutions (see [I]) for L^*, the adjoint of L, in $Q \cap C$ and $R \cap C$ respectively, i.e.

$$L^* G_Q = L^* G_R = \delta(x - z, t - s),\ G_Q = G_R \equiv 0 \text{ when } t > s, \text{ and}$$

$G_Q = 0$ on $\partial Q \cup \partial C$, $G_R = 0$ on $\partial R \cup \partial C$. Given $g \in C_c^\infty(\Phi(D))$, let v_Q^g and v_R^g be G_Q and G_R applied to g respectively. Hence $L^* v_Q^g = L^* v_R^g = g$, v_Q^g and v_R^g vanish on $\partial Q \cup \partial C$ and $\partial R \cup \partial C$ respectively, and $v_Q^g = v_R^g = 0$ for t sufficiently large. We have

$$\int_{Q\cap C} g(x,t) u_Q^f(x,t)\,dxdt = \int_{\partial C} f(x,t)\nu \cdot A(x,t)\nabla_x v_Q^g(x,t)\,dtdS, \text{ and}$$

$$\int_{R\cap C} g(x,t) u_R^f(x,t)\,dxdt = \int_{\partial C} f(x,t)\nu \cdot A(x,t)\nabla_x v_R^g(x,t)\,dtdS,$$

where $\nu = x/|x|$ is the normal to ∂C.

Since we know that $u_Q^f = u_R^f$ on $\Phi(D)$ for all f , it follows that

$$\nu \cdot A(x,t)\nabla_x v_Q^g(x,t) = \nu \cdot A(x,t)\nabla_x v_R^g(x,t)$$

when $(x,t) \in \partial C$. However, by construction $v_Q^g = v_R^g = 0$ on ∂C and $L^*(v_Q^g - v_R^g) = 0$ in $Q \cap R$. Thus we can use the unique continuation argument which showed $u_Q^f = u_R^f$ in $\Phi(D)$ again – this time for L^* instead of L – to conclude that $v_Q^g = v_R^g$ on $\Phi(D)$. Finally, since g was an arbitrary function in $C_c^\infty(\Phi(D))$, we have

$$G_Q(x,t,z,s) = G_R(x,t,z,s) \text{ for all } (x,t),(z,s) \in \Phi(D) \qquad (7).$$

Taking (z,s) in $\Phi(D)$ sufficiently close to (x_0,t_0) in (7) leads to a contradiction to the propagation of singularities for these fundamental solutions. The wave front set of $G_Q(x,t,z,s)$, considered as a distribution in (x,t), consists of all backward null bicharacteristics for L passing over (z,s). Since G_R is constructed with the boundary condition $u = 0$ on ∂R, the corresponding singularities for it are reflected by $\Phi(\Sigma)$ (see [Hö]). We will take (z,s) in D close enough to $\Phi(\Sigma)$ that some singularities will reflect. Then we have a contradiction to the equality of the fundamental solutions. This contradiction completes the proof, and we conclude that $Q = R$.

§**4.2.** In this section we give three results from pseudo-riemannian geometry mentioned in §4.1.

First we show that (x_0,t) will be a time-like curve for L if and only if A is positive definite. Let B be the matrix of the quadratic form $p_2(x,t,\xi,\tau)$ as before. We have $(0,1)\cdot B^{-1}(0,1) = (B^{-1})_{n+1,n+1}$. Since

$$B = \begin{pmatrix} -A & -a \\ -a & 1 \end{pmatrix},$$

the formula for the inverse gives

$$(B^{-1})_{n+1,n+1} = \frac{\det(-A)}{\det(B)} = (-1)^n \frac{\det(A)}{\det(B)}$$

Since B has one positive and n negative eigenvalues $(-1)^n\det(B)$ is positive. Since the quadratic form $w\cdot Aw + (w\cdot a)^2$ is positive definite, A has at most one nonpositive eigenvalue. Thus $(B^{-1})_{n+1,n+1}$ is positive if and only if A is positive definite.

Next we show that, if $v \cdot B^{-1}v > 0$ and w is a nonzero vector such that $v \cdot w = 0$, then $w \cdot Bw < 0$. Note that this implies that a smooth surface of codimension one containing a time-like curve will be time-like at all points on the curve. To see that $w \cdot Bw < 0$ let Z_0 be the normalized eigenvector of B belonging to the positive eigenvalue λ_0. Then we have the orthogonal decomposition of $\mathbb{R}^{n+1}$ into invariant subspaces for B, $\mathbb{R}^{n+1} =< Z_0 > \oplus < Z_0 >^{\perp}$. Let $-C$ denote the restriction of B to $< Z_0 >^{\perp}$. Since B is negative definite on $< Z_0 >^{\perp}$, C is positive definite. We decompose v and w with respect to this orthogonal decomposition as $v = v_0 Z_0 + \tilde{v}$ and $w = w_0 Z_0 + \tilde{w}$. If $w_0 = 0$, we have $w \cdot Bw < 0$. Since $v \cdot w = 0$, $\tilde{v} = 0$ implies $w_0 = 0$, and we have $w \cdot Bw < 0$ again. In the other cases we have

$$v_0^2 w_0^2 = |\tilde{v} \cdot \tilde{w}|^2 = |C^{-1/2}\tilde{v} \cdot C^{1/2}\tilde{w}|^2 \leq (\tilde{v} \cdot C^{-1}\tilde{v})(\tilde{w} \cdot C\tilde{w}) \tag{8}$$

by the Cauchy-Schwarz inequality. Since $v \cdot B^{-1}v > 0$, we have $v_0^2 > \lambda_0 \tilde{v} \cdot C^{-1}\tilde{v}$. Substituting this for v_0^2 in (8), we have $\lambda_0 w_0^2 < \tilde{w} \cdot C\tilde{w}$ which is equivalent to $w \cdot Bw < 0$.

Next we show that one can construct Φ satisfying (i)-(iii) for any Q with complement contained in C and time-like boundary. We will use an analog of the flow $F_{t,s}(x)$ constructed in §3 of [St], adapted to the general hyperbolic operator L. To make all estimates uniform as $|t| \to \infty$ we assume that the coefficients of L and the unit normals to ∂Q have uniformly bounded derivatives, that $A(x,t)$ is uniformly positive definite, and that ∂Q is uniformly time-like. To construct $\Phi(y,t)$ we are going to define a vector field $v(x,t)$ on Q and solve

$$\frac{dF}{dt} = v(F,t), \; F(0,y) = y$$

to obtain a diffeomorphism of Ω_0 onto $\{F(t,y) : y \in \Omega_0\}$. If the vector field $(v(x,t),1)$ is tangent to ∂Q, then $\{F(t,y) : y \in \Omega_0\}$ will be Ω_t (see [H, Chpt. 8] for details). Hence, the mapping

$$\Phi(y,t) = (F(t,y),t)$$

is a diffeomorphism of the cylinder $\Omega_0 \times \mathbb{R}_t$ onto Q. So to satisfy conditions (i)-(iii) we just need to show that we can choose $v(x,t)$ vanishing near ∂C so that $(v(x,t),1)$ is time-like for L. Writing $p_2(x,t,\xi,\tau) = (\xi,\tau) \cdot B(x,t)(\xi,\tau)$ as before, we know that B has one positive and n negative eigenvalues. Hence, letting $\hat{e}_0$ be a unit eigenvector for the positive eigenvalue λ_0, B is negative definite on the orthogonal complement of $\hat{e}_0$. Since we are assuming that ∂Q is time-like, its normal ν satisfies $\nu \cdot B\nu < 0$. We write $\nu = a_0\hat{e}_0 + w$ with $w \cdot \hat{e}_0 = 0$, and normalize ν by requiring $w \cdot Bw = -1$. So with this normalization $\nu \cdot B\nu < 0$ is equivalent to $a_0^2\lambda_0 < 1$. We will look for a time-like vector d tangent to ∂Q in the from

$$d = \hat{e}_0 + z, \; \hat{e}_0 \cdot z = 0.$$

The choice $z = a_0 Bw$, where w is the vector in the representation for ν above, makes $d \cdot \nu = 0$, and turns out to make d time-like as well: we have

$$d \cdot B^{-1}d = \lambda_0^{-1} + a_0^2 Bw \cdot B^{-1}Bw = \lambda_0^{-1} - a_0^2 > 0.$$

We make this choice of $d(x,t)$ on ∂Q. Since the time component of d is uniformly bounded away from zero, we can normalize it to $(v(x,t),1)$ as required.

Since we assume that the derivatives of the coefficients of L and the derivatives of the unit normals to ∂Q are globally bounded, there is a neighborhood

$$\mathcal{N}_\delta = \{(x,t) + s\nu(x,t)), (x,t) \in \partial Q,\ 0 \le s \le \delta\}$$

such that the extension of $v(x,t)$ given by

$$v((x,t) + s\nu(x,t)) = v(x,t)$$

remains in the time-like cone for L. Hence we can smoothly deform v inside the time-like cone to (0,1) inside N_δ. This assures that Φ satisfies (i)-(iii).

§**4.3.** In this section we discuss four examples where moving boundaries are determined uniquely by Cauchy data.

1. Rigidly moving bodies. Take $L = \partial_t^2 - \Delta$ and $\Gamma = \partial C$. We assume that Q is generated by the rotation and translation of a rigid body $\mathcal{B} \subset \{|y| < \rho_0 < \rho\}$. Hence

$$\partial Q = \{(O(t)y + l(t), t) : y \in \partial\mathcal{B}\}.$$

$O(t)$ is the rotation (a real orthogonal matrix of determinant 1), and $l(t)$ is the translation. Without loss of generality we assume $O(0) = I$ and $l(0) = 0$. To keep ∂Q in $\{(x,t) : |x| < \rho\}$ we assume that $|l(t)| < \rho - \rho_0 - \epsilon$ for all t. Finally we assume that the derivatives $O'(t)$ and $l'(t)$ are small enough that we have

$$|O'(t)y| + |l'(t)| \le c < 1$$

when $|y| \le \rho$. The last assumption makes ∂Q uniformly time-like. Note that, when $O(t) \equiv I$, this assumption reduces to $|l'(t)| \le c < 1$ which is necessary as well as sufficient for ∂Q to be uniformly time-like.

In this setting we replace hypothesis (ii) on $\Psi^t(y)$ by

(ii')$\Psi^0(y) = y,\ y \in \Omega_0$, and $\Psi^t(y)$ maps ∂C onto ∂C for all $t \in \mathbb{R}_t$.

Since $O(t)$ is allowed to continue twisting in the same direction for all time – think of a rotation in two space dimensions – it appears difficult to construct Ψ^t satisfying conditions (ii) and (iv) simultaneously. However, since we are taking Γ to be the whole boundary of C, Ψ^t satisfying (i), (ii'), (iii) and (iv) will suffice for the proof of Theorem 4.1. Note that, even though $\Psi^t(y)$ is not the identity on ∂C, equal Cauchy data in the x-coordinates correspond to equal Cauchy data in the y-coordinates. The construction of this Ψ^t can be done as follows.

Note first that $y \to Oy + \beta(|y|)l$ is a diffeomorphism on $|y| > 0$ when $\beta(s)$ is a smooth real-valued function satisfying $|l||\beta'(s)| < 1$. With this in mind choose a smooth $\beta(s)$ satisfying $\beta(s) = 1$ for $s \le \rho_0$ and $\beta(s) = 0$ for $s \ge \rho$ such that $|l(t)||\beta'(s)| < 1$ for all t. This will be possible for some $\epsilon > 0$, since we have $|l(t)| < \rho - \rho_0 - \epsilon$ for all t, as assumed above. One checks easily that with this choice of β

$$\Psi^t(y) = O(t)y + \beta(|y|)\vec{l}(t)$$

satisfies all the requirements.

2. Even periodic motion. Suppose that Q is invariant under both the maps $t \to t+1$ and $t \to -t$. So for each $n \in \mathbb{Z}$ and $t \in [0, 1/2]$ we have $\Omega(n+t) = \Omega(n+1-t)$. This makes it possible to define Ψ as follows: for $t \in [0, 1/2]$ define $\Psi(y,t) = (F(t,y), t)$ where $F(t,y)$ is the mapping from §4.2. For $1/2 \leq t \leq 1$ use $\Psi(y,t) = F(1-t, y)$, and then continue periodically. The resulting mapping satisfies (i)-(iv), but has jumps its time derivative. However, this does not affect the argument. Note that both of the one sided derivatives Φ_t^+ and Φ_t^- are time-like. It is interesting to compare this with the example in §3: the back and forth motion here rules out inaccessible regions.

3. "Slow and uniform" periodic motion. Consider domains $Q \subset \mathbb{R}_x^n \times \mathbb{R}_t$ with ∂Q given by $x(y,t),\ y \in \partial\Omega_0$, where $x(y, t+1) = x(y,t)$. We take $L = \partial_t^2 - \Delta$ and make the following assumptions for all $(y,t) \in \partial\Omega_0 \times \mathbb{R}_t$

$$|x_t(y,t)| \leq \epsilon_0 \text{ and } |D_u x_t(y,t)| \leq \epsilon_0$$

for all directional derivatives D_u with respect to unit vectors tangent to $\partial\Omega_0$ at y. Then, if the constant ϵ_0 is sufficiently small, one can extend x_t smoothly from $\partial\Omega_0 \times \mathbb{R}_t$ to $\{(y,t) \in \mathbb{R}^n \times \mathbb{R}_t : |y| \leq \rho\}$ with the following constraints

(a) $||\frac{\partial x_t}{\partial y}(y,t)|| < 1$. Here $\frac{\partial x_t}{\partial y}$ is the Jacobian matrix, and we are assuming that its matrix norm is less than one.

(b) $x_t(y, t+1) = x_t(y,t)$ and $\int_0^1 x_t(y,t)dt = 0$. Note that $\int_0^1 x_t(y,t)dt = 0$ for $y \in \partial\Omega_0$.

(c) $|x_t(y,t)| < 1$ for all (y,t), and $x_t(y,t) = 0$ for $\rho' \leq |y| \leq \rho$.

Given (a)-(c), we define

$$\Psi(y,t) = y + \int_0^t x_t(y,s)ds.$$

Then $\Psi(y, t+1) = \Psi(y,t)$, and for $t \in [0,1]$ the Jacobian matrix Ψ_y satisfies

$$||\Psi_y(y,t) - I|| \leq || \int_0^t \frac{\partial x_t}{\partial y}(y,s)ds|| < 1.$$

Therefore $\Psi(y,t)$ is a diffeomorphism of Ω_0 onto Ω_t with the properties that ensure that $\Phi(y,t) = (\Psi(y,t), t)$ will satisfy conditions (i)-(iv).

4. Asymptotically stationary motion. Stefanov considered the case of boundaries which are stationary when $|t|$ is sufficiently large in [St]. A generalization of this is the case where the vector field $v(x,t)$ from §4.2 satisfies

$$\int_{\mathbb{R}} \sup_{\{|x| \leq \rho\} \cap \Omega(t)} ||\frac{\partial v}{\partial x}(x,t)||dt < \infty$$

Since the Jacobian of the mapping $F(t,y)$ satisfies

$$\frac{d}{dt}\frac{\partial F}{\partial y} = \frac{\partial v}{\partial x}(F,t)\frac{\partial F}{\partial y},$$

standard stability results (Theorem 1.1, Chpt. X in [Ha]) imply that $||\frac{\partial F}{\partial y}(y,t)||$ is uniformly bounded. Thus, defining $\Phi(y,t) = (F(y,t),t)$ as in §4.2, we have a mapping satisfying (i)-(iv).

The proof that the Jacobian, $\frac{\partial F}{\partial y}(y,t)$, is uniformly bounded is particularly simple in this case: set $\psi(t) = \sup_{\{|x| \leq \rho\}} ||\frac{\partial v}{\partial x}(x,t)||$, and let w be one of the columns in the Jacobian, $w = \frac{\partial F}{\partial y_j}$. Then

$$\frac{1}{2}\frac{d}{dt}|w|^2 = w \cdot \frac{dw}{dt} = w \cdot \frac{\partial v}{\partial x}(F,t)w \leq \psi(t)|w|^2,$$

and for $t > 0$ Gronwall's inequality implies $|w(t)| \leq |w(0)| \exp(\int_0^t \psi(s)ds)$. Similarly for $t < 0$ one has $|w(t)| \leq |w(0)| \exp(\int_t^0 \psi(s)ds)$.

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